

# A Discontinuous Hybrid Mixed Formulation for the 2D Multiscale Porous Media Flow Problem

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## Abstract

In this paper a new hybrid finite element method is introduced, aimed to model multiscale problems with several geometric regions of the domain of interest. In each of these regions porous media fluid flow takes place, but governed by physical parameters at a different scale; additionally, a fluid exchange through contact interfaces occurs between neighboring regions. The well-posedness of the hybrid finite element formulation on bounded simply connected polygonal domains of the plane is presented. Next, the convergence of the discrete solution to the exact solution of the problem is discussed. Finally, the numerical example illustrates the method and experimental rates of convergence.

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## 1. Introduction

Mixed variational formulations are a very important topic of research in applied mathematics. The Babuska-Brezzi theory (see THEOREM 1) is remarkably powerful from the theoretical point of view, however it introduces high complexity in the discrete finite element spaces approximating the solution; this reflects in numerical stability problems (see [1]). The achievements to overcome such difficulty can be on several directions. One of the streams seeks to stabilize the approximation by modifying the bilinear forms involved namely, using symmetric properties of the tensors as in [2, 3] or including terms in the bilinear forms in a “balanced” way as in [1, 4]. As this technique has proved to be fruitful and rich in terms of the possibilities to stabilize the forms of interest, some other aspects arise by itself, such as the discussion of minimal stabilisation procedures (see [5]), or the a-priori, a-posteriori error analysis for these new scheme (see [3]). A second approach uses discontinuous Galerkin finite elements (DG). The DG methods have several advantages and goals, some of these are: addressing non-conformality in a more flexible way, treating stability issues due to coupling constraints (demanding regularity in the discrete spaces), and computing in a more accurate way the physical quantity that is known to be predominant in specific subregions. The latter is attained in two ways, by local refining of the mesh and by approximating polynomial spaces; see [6, 7] for a unified vision of the DG Methods.

All the aforementioned works, whichever the problem they may be analyzing (elasticity, heat diffusion, free flow, Darcy flow, etc), treat separately the primal and dual mixed formulations (see [8, 9, 10]). The present paper is focused on using simultaneously both fundamental versions for the treatment of multiscale problems in Darcy flow (see PROBLEM (1)), it is therefore a mixed-mixed formulation; in a way this article is the numerical implementation of the formulation introduced in [11] (see also [12, 13] for related formulations). The stability aspects become particularly critical when dealing with multiscale problems, as the presence of physical coefficients of different orders adds up to the built-in complexity of the mixed variational formulations (coefficient  $a(\cdot)$  in

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PROBLEM (1)). The mixed-mixed formulation tackles this issue by removing coupling constraints from the discrete trial spaces while satisfying them only on the solution i.e., the continuous formulation replaces *strong coupling* conditions by *weak coupling* conditions (see EQUATIONS (7) and PROBLEM (14)). Replacing the nature of the coupling conditions is a strategy already used in DG methods using penalization techniques; however, this is done only on the discrete version, while the continuous formulation still relies on strong coupling conditions. The latter is because, in the Darcy flow problem, while the primal mixed formulation can introduce weak coupling conditions on the normal flow exchange, the normal stress has to stay continuous. In contrast, the dual mixed formulation can introduce weak coupling conditions for the normal stress balance, but it requires the normal flow exchange to be continuous. The continuity constraints of the classical mixed formulations reflect later on, in the deep discussions of convergence present in the DG methods.

Another advantage of the discrete mixed-mixed formulation we are to introduce in this work is that, according to the regions, the predominant effect can be chosen to be modeled with the discrete space holding the sense of continuity, while the secondary effect is modeled with the discontinuous space. In the case of Darcy flow, the pressure is the dominant effect in regions of low permeability, while the flow velocity is the predominant one in regions of high permeability (see FIGURE 4 (a)). This concept has already arisen naturally in previous DG methods coupling advection with diffusion phenomena, due to the discrete spaces involved in the formulations, see [14]. To the author's best knowledge there is no precedent for having this level of flexibility in the analysis of coupling fluid flow phenomena, as the literature analyzing multiscale flow is mainly focused in coupling Stokes flow with Darcy flow, see [15, 16, 17, 18].

The proposed model is to analyze a variation of the classic porous media problem on a connected bounded open region  $\Omega \subset \mathbb{R}^2$ , i.e.,

$$a(\cdot)\mathbf{u} + \nabla p + \mathbf{g} = 0, \quad (1a)$$

$$\nabla \cdot \mathbf{u} = F \quad \text{in } \Omega. \quad (1b)$$

$$p = 0 \quad \text{on } \Gamma_d, \quad (1c)$$

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \Gamma_f \stackrel{\text{def}}{=} \partial\Omega - \Gamma_d; \quad (1d)$$

more specifically, when  $\Omega$  is partitioned in two subdomains  $\Omega_1, \Omega_2$  such that  $a(\cdot)|_{\Omega_1} = O(1)$  and  $a(\cdot)|_{\Omega_2} = O(\epsilon)$  for  $\epsilon > 0$  small (see FIGURE 1). In this context the continuity of the solution  $[\mathbf{u}, p]$  across the interface between  $\Omega_1$  and  $\Omega_2$  becomes a liability from the numerical point of view. Therefore, if it is possible to estimate a-priori, the magnitude of change that the solution will experience from one subdomain to the other, it is a more strategic approach to artificially introduce a discontinuity across the interface, satisfying a balance/coupling condition for both, normal flux and normal stress: see EQUATIONS (7). As mentioned above, these exchange conditions will be introduced weakly in the formulation allowing full decoupling of the underlying function spaces. Moreover, the trial spaces require that the pressure  $q$  is only square integrable on one side of the interface  $L^2$  while it belongs to  $H^1$  on the other side of the interface (see FIGURE 4 (a)), such discontinuity on the test spaces is ideal to handle discontinuities on the normal stress across the interface. The analogous takes place on the velocities modeling spaces, here the test functions  $\mathbf{v}$  belong to  $\mathbf{H}_{\text{div}}$  on one side of the interface while only square integrable  $\mathbf{L}^2$  on the other (see FIGURE 4 (b)). Again, this scenario will be ideal for discontinuities of normal flux across the interface.

We close this section introducing the general notation. In the present work vectors are denoted by boldface letters as are vector-valued functions and corresponding function spaces. The symbols  $\nabla$  and  $\nabla \cdot$  represent the gradient and divergence operators respectively. The dimension is indicated by  $N$  which will be equal to 2 or 3 depending on the context. Given a function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  then  $\int_{\mathcal{M}} f dS$  denotes the integral on the  $N - 1$  dimensional manifold  $\mathcal{M} \subseteq \mathbb{R}^N$ . Analogously,  $\int_A f d\mathbf{x}$  stands for the integral in the set  $A \subseteq \mathbb{R}^N$ ; whenever the context is clear we simply write  $\int_A f$ . Given an open set  $G$  of  $\mathbb{R}^N$ , the symbols  $\|\cdot\|_{0,G}$ ,  $\|\cdot\|_{1,G}$ ,  $\|\cdot\|_{1/2,\partial G}$ ,  $\|\cdot\|_{-1/2,\partial G}$  and  $\|\cdot\|_{\mathbf{H}_{\text{div}}(G)}$  denote the  $L^2(G)$ ,  $H^1(G)$ ,  $H^{1/2}(\partial G)$ ,  $H^{-1/2}(\partial G)$  and  $\mathbf{H}_{\text{div}}(G)$  norms respectively, while  $|M|$  represents the Lebesgue measure of  $G$  in  $\mathbb{R}$ ,  $\mathbb{R}^2$  or  $\mathbb{R}^3$  depending on the context.

## 2. Preliminaries

### 2.1. Geometric Setting

In this section we set the conditions on the domain of reference as well as its gridding.

**Definition 1.** Given a bounded open set  $\omega$  in  $\mathbb{R}^2$  we will say that a **bipartite map** is a finite collection of connected open subsets  $\mathcal{G} = \{G_n : 1 \leq n \leq N\}$  such that

- (i) If  $n \neq k$  then  $G_n \cap G_k = \emptyset$ .
- (ii) The union satisfies  $\left| \omega - \bigcup_{i=1}^N G_n \right| = 0$  and  $\text{cl}(\omega) = \bigcup_{i=1}^N \text{cl}(G_n)$ .
- (iii) The collection  $\mathcal{G} = \{G_n : 1 \leq n \leq N\}$  is partitioned in two subcollections  $\mathcal{G}_1 = \{G_i^1 : 1 \leq i \leq I\}$  and  $\mathcal{G}_2 = \{G_j^2 : 1 \leq j \leq J\}$  such that
  - a)  $\{G_n : 1 \leq n \leq N\} = \{G_i^1 : 1 \leq i \leq I\} \cup \{G_j^2 : 1 \leq j \leq J\}$ .
  - b) If  $i \neq k$  then  $|\partial G_i^1 \cap \partial G_k^1| = 0$ .
  - c) If  $j \neq \ell$  then  $|\partial G_j^2 \cap \partial G_\ell^2| = 0$ .

The collections  $\mathcal{G}_1, \mathcal{G}_2$  are said to be the **bipartition** or the **bi-coloring** of the map.

**Hypothesis 1.** The domain of interest  $\Omega$  is a polygonal, bounded, connected region of the plane and, it satisfies that

- (i) It has a bipartite map  $\mathcal{G} = \{G_n : 1 \leq n \leq N\}$  such that  $G_n$  is a polygon for each  $n = 1, \dots, N$ .
- (ii) If  $\mathcal{G}_1, \mathcal{G}_2$  is the bipartition of the map  $\mathcal{G}$  then  $\text{cl}[\bigcup\{L : L \in \mathcal{G}_1\}]$  and  $\text{cl}[\bigcup\{L : L \in \mathcal{G}_2\}]$  are connected.

An example of bipartite map is depicted in FIGURE 1 (a), together with some other concepts introduced in the following definition.

**Definition 2.** Let  $\Omega$  satisfy HYPOTHESIS 1 and let  $\mathcal{G} = \{G_n : 1 \leq n \leq N\}$  be its bipartite map with  $\mathcal{G}_1, \mathcal{G}_2$  the map bipartition.

- (i) For each polygon  $K \in \mathcal{G}$  denote by  $\hat{\nu}$  the outer normal vector to its boundary  $\partial K$ .
- (ii) For each polygon  $K \in \mathcal{G}$  define  $\hat{\mathbf{n}}$  by

$$\hat{\mathbf{n}}(\vec{x}) \stackrel{\text{def}}{=} \begin{cases} \hat{\nu}(\vec{x}) & K \in \mathcal{G}_1 \text{ and } \vec{x} \in \partial K, \\ -\hat{\nu}(\vec{x}) & K \in \mathcal{G}_2 \text{ and } \vec{x} \in \partial K \cap \Omega, \\ \hat{\nu}(\vec{x}) & K \in \mathcal{G}_2 \text{ and } \vec{x} \in \partial K \cap \partial\Omega. \end{cases} \quad (2)$$

- (iii) Define  $\Omega_1 \stackrel{\text{def}}{=} \bigcup\{L : L \in \mathcal{G}_1\}$  and  $\Omega_2 \stackrel{\text{def}}{=} \bigcup\{M : M \in \mathcal{G}_2\}$ .

- (iv) Denote by  $\Gamma \stackrel{\text{def}}{=} \bigcup\{\partial K : K \in \mathcal{G}\} - \partial\Omega$ , the **interface** of the domain.

Next, we define the type of grids that will be considered in this work, see FIGURE 1 (b) for a simple example.

**Definition 3.** Let  $\Omega$  be as in DEFINITION 2 above, then

- (i) A triangulation  $\mathcal{T}$  of the domain  $\Omega$  is said to be **consistent** with the map  $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$  if for each triangle  $K \in \mathcal{T}$ , it holds that  $K \cap \Omega_1 = \emptyset$  or  $K \cap \Omega_2 = \emptyset$ . Equivalently,  $\mathbb{1}_{K \cap \Omega_1}(\cdot) \mathbb{1}_{K \cap \Omega_2}(\cdot) = 0$ .
- (ii) Given two triangulations  $\mathcal{T}'$  and  $\mathcal{T}$  of the domain  $\Omega$ , we say that  $\mathcal{T}'$  is a **refinement** of  $\mathcal{T}$ , denoted by  $\mathcal{T}' \leq \mathcal{T}$ , if for each element  $K' \in \mathcal{T}'$  there exists a triangle  $K \in \mathcal{T}$  such that  $K' \subseteq K$ .
- (iii) A sequence  $\{\mathcal{T}^h : h > 0\}$  is said to be **monotone** if  $h' < h$  implies that  $\mathcal{T}^{h'} \leq \mathcal{T}^h$ .

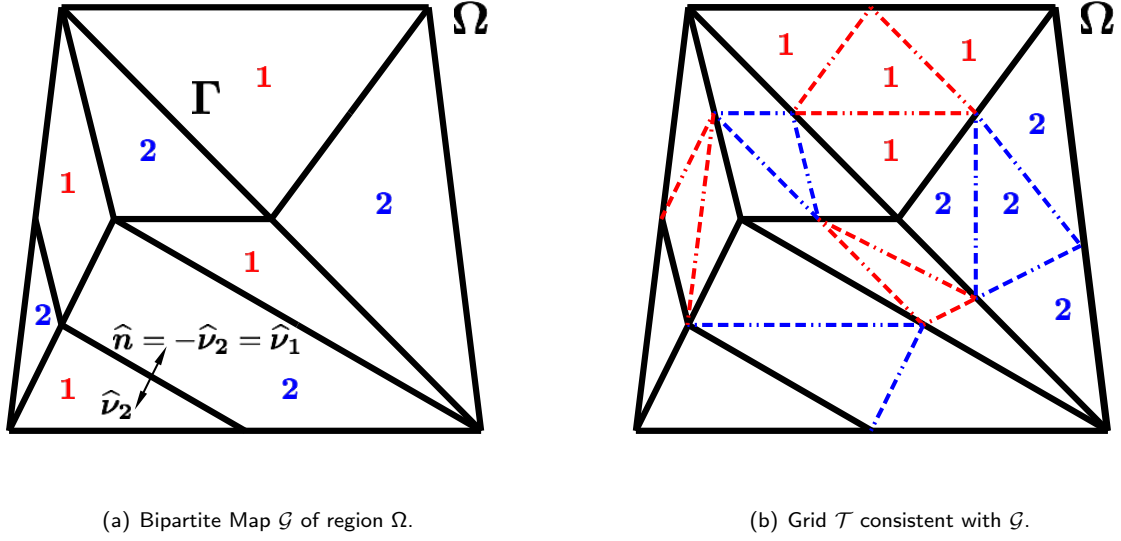


Figure 1: Figure (a) depicts a bipartite map  $\mathcal{G}$  example for a given region  $\Omega$ . Subregions belonging to  $\mathcal{G}_i$  have been labeled with  $i$  for  $i = 1, 2$ . The vector  $\hat{\mathbf{n}}$  and the outer normal vectors  $\hat{\nu}_1, \hat{\nu}_2$  are illustrated for a couple of neighboring elements belonging to  $\mathcal{G}_1$  and  $\mathcal{G}_2$  respectively. Figure (b) depicts an example of grid  $\mathcal{T}$  consistent with the map  $\mathcal{G}$ . Some of the triangles belonging to  $\mathcal{T}_1$  and  $\mathcal{T}_2$  have been labeled with 1 and 2 respectively.

## 2.2. The Strong Problem and its Continuous Weak Formulation

We begin this section recalling the general abstract setting to be used in this article. Let  $\mathbf{X}$  and  $\mathbf{Y}$  be Hilbert spaces and let  $\mathcal{A} : \mathbf{X} \rightarrow \mathbf{X}'$ ,  $\mathcal{B} : \mathbf{X} \rightarrow \mathbf{Y}'$  and  $\mathcal{C} : \mathbf{Y} \rightarrow \mathbf{Y}'$  be continuous linear operators, we are to work on the following problem

$$\begin{aligned} \text{Find a pair } (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y} : \quad & \mathcal{A}\mathbf{x} + \mathcal{B}'\mathbf{y} = F_1 \quad \text{in } \mathbf{X}', \\ & -\mathcal{B}\mathbf{x} + \mathcal{C}\mathbf{y} = F_2 \quad \text{in } \mathbf{Y}', \end{aligned} \quad (3)$$

where  $F_1 \in \mathbf{X}'$  and  $F_2 \in \mathbf{Y}'$ . The following is a well-known result [19].

**Theorem 1.** Assume that the linear operators  $\mathcal{A} : \mathbf{X} \rightarrow \mathbf{X}'$ ,  $\mathcal{B} : \mathbf{X} \rightarrow \mathbf{Y}'$ ,  $\mathcal{C} : \mathbf{Y} \rightarrow \mathbf{Y}'$  are continuous and

(i)  $\mathcal{A}$  is non-negative and  $\mathbf{X}$ -coercive on  $\ker(\mathcal{B})$ .

(ii)  $\mathcal{B}$  satisfies the inf-sup condition

$$\inf_{\mathbf{y} \in \mathbf{Y}} \sup_{\mathbf{x} \in \mathbf{X}} \frac{|\mathcal{B}\mathbf{x}(\mathbf{y})|}{\|\mathbf{x}\|_{\mathbf{X}} \|\mathbf{y}\|_{\mathbf{Y}}} > 0. \quad (4)$$

(iii)  $\mathcal{C}$  is non-negative symmetric.

Then for every  $F_1 \in \mathbf{X}'$  and  $F_2 \in \mathbf{Y}'$  the PROBLEM (3) has a unique solution in  $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$ ; additionally it satisfies the estimate

$$\|\mathbf{x}\|_{\mathbf{X}} + \|\mathbf{y}\|_{\mathbf{Y}} \leq c (\|F_1\|_{\mathbf{X}'} + \|F_2\|_{\mathbf{Y}'}). \quad (5)$$

Next, we present the strong problem to be approximated. Given a region  $\Omega$  verifying HYPOTHESIS 1, we introduce the following generalization of the Darcy flow PROBLEM (1).

$$a(\cdot) \mathbf{u}_1 + \nabla p_1 + \mathbf{g} = 0, \quad (6a) \quad a(\cdot) \mathbf{u}_2 + \nabla p_2 + \mathbf{g} = 0, \quad (6d)$$

$$\nabla \cdot \mathbf{u}_1 = F \quad \text{in } \Omega_1. \quad (6b) \quad \nabla \cdot \mathbf{u}_2 = F \quad \text{in } \Omega_2. \quad (6e)$$

$$p_1 = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega. \quad (6c) \quad \mathbf{u}_2 \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega. \quad (6f)$$

Endowed with the following interface exchange balance conditions

$$p_2 - p_1 = f_\Sigma, \quad (7a)$$

$$\mathbf{u}_1 \cdot \hat{\mathbf{n}} - \mathbf{u}_2 \cdot \hat{\mathbf{n}} = \beta(\cdot) p_2 + f_{\hat{\mathbf{n}}}. \quad (7b)$$

The problem above, allows discontinuity jumps of discontinuity across the interface  $\Gamma$ , due to the forcing terms in the normal stress (7a) and normal flux balance conditions (7b). The coefficients  $a(\cdot)$ ,  $\beta(\cdot)$  are nonnegative and they stand for the medium resistance to the fluid flow and the interface storage rate, respectively.

In order to introduce the modeling spaces to be used in the weak variational formulation, first notice that  $\{L : L \in \mathcal{T}_1\}$ ,  $\{M : M \in \mathcal{T}_2\}$  are the simply connected components of  $\Omega_1$  and  $\Omega_2$  respectively. Then,

$$\mathbf{H}_{\text{div}}(\Omega_1) = \bigoplus_{L \in \mathcal{T}_1} \mathbf{H}_{\text{div}}(L), \quad H^1(\Omega_2) = \bigoplus_{M \in \mathcal{T}_2} H^1(M).$$

The following space is introduced in order to couple adequately, the action of the pressure traces in the variational formulation

$$\begin{aligned} E(\Omega_2) &\stackrel{\text{def}}{=} \{q \in H^1(\Omega_2) : q \mathbb{1}_{\partial M \cap \partial L} \in H^{1/2}(\partial L) \text{ for all } (L, M) \in \mathcal{G}_1 \times \mathcal{G}_2\} \\ &= \{q \in H^1(\Omega_2) : q \mathbb{1}_\Gamma \in H^{1/2}(\Gamma)\}. \end{aligned} \quad (8)$$

We endow  $E(\Omega_2)$  with the  $H^1(\Omega_2)$  inner product. It is direct to see that  $E(\Omega_2)$  is a closed subspace of  $H^1(\Omega_2)$  and consequently a Hilbert space. Also define

$$\mathbf{V}(\Omega_2) \stackrel{\text{def}}{=} \{\mathbf{v} \in \mathbf{L}^2(\Omega_2) : \mathbf{v}_2 = \nabla q_2 \text{ for some } q_2 \in E(\Omega_2)\} = \nabla(E(\Omega_2)), \quad (9)$$

endowed with the  $\mathbf{L}^2(\Omega_2)$  inner product. Next we recall a necessary result.

**Lemma 2.** *Let  $E(\Omega_2)$  and  $\mathbf{V}(\Omega_2)$  be as defined in (8), (9) respectively; define*

$$E_0(\Omega_2) \stackrel{\text{def}}{=} \left\{ q_2 \in E(\Omega_2) : \int_{\Omega_2} q_2 = 0 \right\}. \quad (10)$$

Then,

(i) *There exists a constant  $C > 0$  depending only on the domain  $\Omega_2$  such that*

$$\|r_2\|_{1,\Omega_2} \leq C \|\nabla r_2\|_{0,\Omega_2}, \quad \text{for all } r_2 \in H. \quad (11)$$

(ii) *The space  $\mathbf{V}(\Omega_2)$  is Hilbert.*

PROOF. See LEMMA 21 in [11]. □

Now we are ready to introduce the functional setting of the problem, define

$$\mathbf{X} \stackrel{\text{def}}{=} \mathbf{H}_{\text{div}}(\Omega_1) \times E(\Omega_2). \quad (12a)$$

$$\mathbf{Y} \stackrel{\text{def}}{=} \mathbf{V}(\Omega_2) \times L^2(\Omega_1). \quad (12b)$$

Endowed with their natural norms

$$\|[\mathbf{v}_1, q_2]\|_{\mathbf{X}} \stackrel{\text{def}}{=} \left\{ \|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 + \|q_2\|_{H^1(\Omega_2)}^2 \right\}^{\frac{1}{2}}, \quad (12c)$$

$$\|[\mathbf{v}_2, q_1]\|_{\mathbf{Y}} \stackrel{\text{def}}{=} \left\{ \|\mathbf{v}_2\|_{\mathbf{L}^2(\Omega_2)}^2 + \|q_1\|_{L^2(\Omega_1)}^2 \right\}^{\frac{1}{2}}. \quad (12d)$$

**Remark 1.** (i) Clearly  $\mathbf{X}$  is a Hilbert space, in order to see that  $\mathbf{Y}$  is a Hilbert space see LEMMA 2 above and/or LEMMA 21 in [11].

(ii) In order to avoid heavy notation, from now on the following notational convention will be adopted

$$\int_{\Gamma} (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) q_2 dS \stackrel{\text{def}}{=} \langle \mathbf{v}_1 \cdot \hat{\mathbf{n}}, q_2 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \quad (13)$$

From now on, we assume that  $F \in L^2(\Omega)$ ,  $\mathbf{g} \in \mathbf{L}^2(\Omega_2)$ ,  $f_{\Sigma} \in H^{1/2}(\Gamma)$  and  $f_{\hat{\mathbf{n}}} \in H^{-1/2}(\Gamma)$ . Finally, the mixed-mixed formulation for the PROBLEM (6) with interface balance conditions (7) is given by

$$\begin{aligned} \text{Find } ([\mathbf{u}_1, p_2], [\mathbf{u}_2, p_1]) \in \mathbf{X} \times \mathbf{Y} : \quad & \int_{\Omega_1} a \mathbf{u}_1 \cdot \mathbf{v}_1 + \int_{\Gamma} \beta p_2 q_2 dS - \int_{\Gamma} (\mathbf{u}_1 \cdot \hat{\mathbf{n}}) q_2 dS + \int_{\Gamma} p_2 (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS \\ & - \int_{\Omega_1} p_1 \nabla \cdot \mathbf{v}_1 - \int_{\Omega_2} \mathbf{u}_2 \cdot \nabla q_2 = \int_{\Omega_2} F q_2 - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}_1 + \int_{\Gamma} f_{\Sigma} (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS - \int_{\Gamma} f_{\hat{\mathbf{n}}} q_2 dS, \end{aligned} \quad (14a)$$

$$\begin{aligned} \int_{\Omega_1} \nabla \cdot \mathbf{u}_1 q_1 + \int_{\Omega_2} \nabla p_2 \cdot \mathbf{v}_2 + \int_{\Omega_2} a \mathbf{u}_2 \cdot \mathbf{v}_2 &= \int_{\Omega_1} F q_1 - \int_{\Omega_2} \mathbf{g} \cdot \mathbf{v}_2, \\ &\text{for all } ([\mathbf{v}_1, q_2], [\mathbf{v}_2, q_1]) \in \mathbf{X} \times \mathbf{Y}. \end{aligned} \quad (14b)$$

Define the operators  $\mathcal{A} : \mathbf{X} \rightarrow \mathbf{X}'$ ,  $\mathcal{B} : \mathbf{X} \rightarrow \mathbf{Y}'$  and  $\mathcal{C} : \mathbf{Y} \rightarrow \mathbf{Y}'$  by

$$\mathcal{A}[\mathbf{v}_1, q_2], ([\mathbf{w}_1, r_2]) \stackrel{\text{def}}{=} \int_{\Omega_1} a \mathbf{v}_1 \cdot \mathbf{w}_1 + \int_{\Gamma} \beta q_2 r_2 dS - \int_{\Gamma} (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) r_2 dS + \int_{\Gamma} q_2 (\mathbf{w}_1 \cdot \hat{\mathbf{n}}) dS, \quad (15a)$$

$$\mathcal{B}[\mathbf{v}_1, q_2], ([\mathbf{w}_2, r_1]) \stackrel{\text{def}}{=} \int_{\Omega_1} \nabla \cdot \mathbf{v}_1 r_1 + \int_{\Omega_2} \nabla q_2 \cdot \mathbf{w}_2, \quad (15b)$$

$$\mathcal{C}[\mathbf{v}_2, q_1]([\mathbf{w}_2, r_1]) \stackrel{\text{def}}{=} \int_{\Omega_2} a \mathbf{v}_2 \cdot \mathbf{w}_2. \quad (15c)$$

Hence, the PROBLEM (14) is equivalent to

$$\begin{aligned} \text{Find a pair } ([\mathbf{u}_2, p_1], [\mathbf{u}_1, p_2]) \in \mathbf{X} \times \mathbf{Y} : \quad & \mathcal{A}[\mathbf{u}_2, p_1] + \mathcal{B}'[\mathbf{u}_1, p_2] = F_1 \quad \text{in } \mathbf{X}', \\ & -\mathcal{B}[\mathbf{u}_2, p_1] + \mathcal{C}[\mathbf{u}_1, p_2] = F_2 \quad \text{in } \mathbf{Y}'. \end{aligned} \quad (16)$$

Here  $F_1 \in \mathbf{X}'$  and  $F_2 \in \mathbf{Y}'$  are the functionals defined by the right hand side of (14a) and (14b) respectively. In order to satisfy the required ellipticity conditions for the operator  $\mathcal{A}$ , some extra hypotheses on the coefficients become necessary.

**Hypothesis 2.** It will be assumed that coefficients of storage exchange  $\beta : \Gamma \rightarrow [0, \infty)$  and porous medium resistance  $a : \Omega \rightarrow (0, \infty)$ , satisfy that  $\beta \in L^\infty(\Gamma)$ ,  $\|\beta \mathbf{1}_\Gamma\|_{L^1(\Gamma)} > 0$  and  $a \in L^\infty(\Omega)$ ,  $\|\frac{1}{a}\|_{L^\infty(\Omega)} > 0$  respectively.

**Theorem 3.** Let  $\Omega$  be a polygonal region and let  $\mathcal{G}$  be a bipartite map, then if the HYPOTHESIS 2 is satisfied, the PROBLEM (14) is well-posed.

PROOF. See THEOREM 25 in [11].  $\square$

We close this section recalling the next result on recovering the strong problem from the weak variational formulation (14).

**Theorem 4.** The solution of the weak variational PROBLEM (14) is a strong solution of the PROBLEM (6) with the forcing gravitation term  $\mathbf{g}$  in the EQUATION (6a) replaced by  $P\mathbf{g}$ , which denotes its orthogonal projection onto the space  $\mathbf{V}(\Omega_2)$ . In particular if  $\mathbf{g}\mathbf{1}_{\Omega_2} \in \mathbf{V}(\Omega_2)$  the weak solution is exactly the strong solution.

PROOF. See THEOREM 26 in [11].  $\square$

### 3. The Discretization of the Problem

In this section we present a viable discretization of the Problem (14) in the two dimensional case, from the theoretical point of view. We start introducing the discrete function spaces, we will denote by  $P_\ell(K)$  the polynomials of order  $\ell$  on the triangle  $K$  and  $\mathbf{P}_\ell(K) = (P_\ell(K))^2$ . As usual,  $\mathbf{RT}_\ell(K)$  indicates the Raviart-Thomas finite element of degree  $\ell$  on the triangle  $K$ . From now on it will be assumed that the domain  $\Omega$  satisfies HYPOTHESIS 1 and that any triangulation  $\mathcal{T}$  of analysis is consistent with the map  $\mathcal{G}$ , as introduced in DEFINITION 3. Hence, for a fixed consistent triangulation  $\mathcal{T}$  with size  $h \stackrel{\text{def}}{=} \max\{\text{diameter}(K) : K \in \mathcal{T}\}$  we denote

$$\mathbf{RT}_0(\Omega_1, \mathcal{T}) \stackrel{\text{def}}{=} \{\mathbf{v} \in \mathbf{H}_{\text{div}}(\Omega_1) : \mathbf{v}|_K \in \mathbf{RT}_0(K), \text{ for all } K \in \mathcal{T}, K \subseteq \Omega_1\}, \quad (17a)$$

$$\mathcal{Q}(\Omega_2, \mathcal{T}) \stackrel{\text{def}}{=} \{q_2 \in H^1(\Omega_2) : q_2 = \xi|_{\Omega_2} \text{ for some } \xi \in H^1(\Omega) \text{ and } q_2|_K \in P_1(K), \text{ for all } K \in \mathcal{T}, K \subseteq \Omega_2\}, \quad (17b)$$

$$\nabla \mathcal{Q}(\Omega_2, \mathcal{T}) \stackrel{\text{def}}{=} \{\mathbf{v}_2 \in \mathbf{P}_0(\Omega_2) : \mathbf{v}_2 = \nabla q_2 \text{ for some } q_2 \in \mathcal{Q}(\Omega_2, \mathcal{T})\}, \quad (17c)$$

$$\mathcal{Q}(\Omega_1, \mathcal{T}) \stackrel{\text{def}}{=} \{q_1 \in L^2(\Omega_1) : q_1|_K \in P_0(K), \text{ for all } K \in \mathcal{T}, K \subseteq \Omega_1\}. \quad (17d)$$

If the triangulation  $\mathcal{T}$  is clear from the context, we simply write  $\mathbf{RT}_0(\Omega_1) = \mathbf{RT}_0(\Omega_1, \mathcal{T})$ ,  $\nabla \mathcal{Q}(\Omega_2) = \nabla \mathcal{Q}(\Omega_2, \mathcal{T})$  and  $\mathcal{Q}(\Omega_\ell) = \mathcal{Q}(\Omega_\ell, \mathcal{T})$  for  $\ell = 1, 2$ . Notice that  $\mathbf{RT}_0(\Omega_1) \times \mathcal{Q}(\Omega_2) \subseteq \mathbf{X}$  and  $\nabla \mathcal{Q}(\Omega_2) \times \mathcal{Q}(\Omega_1) \subseteq \mathbf{Y}$ . Define the following discrete spaces

$$\mathbf{X}_h \stackrel{\text{def}}{=} \mathbf{RT}_0(\Omega_1) \times \mathcal{Q}(\Omega_2), \quad (18a)$$

$$\mathbf{Y}_h \stackrel{\text{def}}{=} \nabla \mathcal{Q}(\Omega_2) \times \mathcal{Q}(\Omega_1), \quad (18b)$$

endowed  $\mathbf{X}_h, \mathbf{Y}_h$  with the norms  $\|\cdot\|_{\mathbf{X}}$  and  $\|\cdot\|_{\mathbf{Y}}$  respectively. The discrete operators  $\mathcal{A}_h : \mathbf{X}_h \rightarrow \mathbf{X}'_h$ ,  $\mathcal{B}_h : \mathbf{X}_h \rightarrow \mathbf{Y}'_h$  and  $\mathcal{C}_h : \mathbf{Y}_h \rightarrow \mathbf{Y}'_h$  are defined by the respective restriction of the operators  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  introduced in (15a), (15b) and (15c) i.e.,

$$\mathcal{A}_h[\mathbf{v}_1, q_2]([\mathbf{w}_1, r_2]) \stackrel{\text{def}}{=} \mathcal{A}[\mathbf{v}_1, q_2]([\mathbf{w}_1, r_2]), \quad \text{for all } [\mathbf{v}_1, q_2], [\mathbf{w}_1, r_2] \in \mathbf{X}_h. \quad (19a)$$

$$\mathcal{B}_h[\mathbf{v}_1, q_2], ([\mathbf{w}_2, r_1]) \stackrel{\text{def}}{=} \mathcal{B}[\mathbf{v}_1, q_2]([\mathbf{w}_2, r_1]), \quad \text{for all } [\mathbf{v}_1, q_2] \in \mathbf{X}_h, [\mathbf{w}_2, r_1] \in \mathbf{Y}_h. \quad (19b)$$

$$C_h[\mathbf{v}_2, q_1]([\mathbf{w}_2, r_1]) \stackrel{\text{def}}{=} \mathcal{C}[\mathbf{v}_2, q_1]([\mathbf{w}_2, r_1]), \quad \text{for all } [\mathbf{v}_2, q_1], [\mathbf{w}_2, r_1] \in \mathbf{Y}_h. \quad (19c)$$

The discretization of PROBLEM (16) is given by

$$\begin{aligned} \text{Find a pair } ([\mathbf{u}_2^h, p_1^h], [\mathbf{u}_1^h, p_2^h]) \in \mathbf{X}_h \times \mathbf{Y}_h : \quad & \mathcal{A}_h[\mathbf{u}_2^h, p_1^h] + \mathcal{B}_h'[\mathbf{u}_1^h, p_2^h] = F_1 \quad \text{in } \mathbf{X}_h', \\ & -\mathcal{B}_h[\mathbf{u}_2^h, p_1^h] + C_h[\mathbf{u}_1^h, p_2^h] = F_2 \quad \text{in } \mathbf{Y}_h', \end{aligned} \quad (20)$$

where  $F_1 \in \mathbf{X}_h'$  and  $F_2 \in \mathbf{Y}_h'$  are known functionals. We are to prove that the PROBLEM (20) above is well-posed, verifying that the operators  $\mathcal{A}_h$ ,  $\mathcal{B}_h$  and  $C_h$  satisfy the hypotheses of THEOREM 1. Before proving the inf-sup condition of the operator  $\mathcal{B}_h$  we recall a well-known result

**Theorem 5.** *Let  $\mathbf{RT}_0(\Omega_1, \mathcal{T})$ ,  $\mathcal{Q}(\Omega_1, \mathcal{T})$  be as defined in (17a) and (17d) respectively. Then, for every  $q_1 \in \mathcal{Q}(\Omega_1, \mathcal{T})$  there exist  $\mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1, \mathcal{T})$  and a constant  $C > 0$  depending only on  $\Omega_1$  such that  $\nabla \cdot \mathbf{v}_1 = q_1$  and  $\|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq C \|q_1\|_{L^2(\Omega_1)}$ .*

PROOF. See LEMMA 5.4, Chapter III, pg 151 in [20].  $\square$

**Lemma 6.** *The operator  $\mathcal{B}_h : \mathbf{X}_h \rightarrow \mathbf{Y}_h'$  defined in EQUATION (19b) is continuous and satisfies the inf-sup condition i.e., there exists a constant  $C > 0$  depending only on the map  $\mathcal{G}$  such that for every  $[\mathbf{w}_2, r_1] \in \mathbf{Y}_h$  there exists  $[\mathbf{v}_1, q_2] \in \mathbf{X}_h$  satisfying*

$$\mathcal{B}_h[\mathbf{v}_1, q_2]([\mathbf{w}_2, r_1]) \geq C \|[\mathbf{v}_1, q_2]\|_{\mathbf{X}_h} \|[\mathbf{w}_2, r_1]\|_{\mathbf{Y}_h}. \quad (21)$$

Moreover, the constant  $C > 0$  is independent from  $[\mathbf{w}_2, r_1]$  and the triangulation  $\mathcal{T}$ .

PROOF. The continuity of  $\mathcal{B}_h$  follows from the continuity of  $\mathcal{B}$ . Now fix  $[\mathbf{w}_2, r_1] \in \mathbf{Y}_h$ , due to THEOREM 5 there exists  $\mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1)$  such that  $\nabla \cdot \mathbf{v}_1 = r_1$  and  $\|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq C \|r_1\|_{L^2(\Omega_1)}$ , with  $C > 0$  depending only on the domain  $\Omega_1$ .

Next, by definition of  $\nabla \mathcal{Q}(\Omega_2)$  there must exist  $\eta \in \mathcal{Q}(\Omega_2)$  such that  $\nabla \eta = \mathbf{w}_2$ . Define  $q_2 \stackrel{\text{def}}{=} \eta - \frac{1}{|\Omega_2|} \int_{\Omega_2} \eta$ , clearly  $q_2 \in E_0(\Omega_2) \cap \mathcal{Q}(\Omega_2)$  and due to the INEQUALITY (11), it holds that  $\|q_2\|_{1, \Omega_2} \leq C \|\mathbf{w}_2\|_{0, \Omega_2}$  with  $C > 0$  depending only on the domain  $\Omega_2$ .

Consequently, the pair  $[\mathbf{v}_1, q_2]$  belongs to  $\mathbf{X}_h$  and satisfies  $\|[\mathbf{v}_1, q_2]\|_{\mathbf{X}} \leq C \|[\mathbf{w}_2, r_1]\|_{\mathbf{Y}}$  with  $C > 0$  adequate depending only on the domain  $\Omega$ . Therefore,

$$\mathcal{B}_h[\mathbf{v}_1, q_2]([\mathbf{w}_2, r_1]) = \|[\mathbf{w}_2, r_1]\|_{\mathbf{Y}}^2 \geq \frac{1}{C} \|[\mathbf{v}_1, q_2]\|_{\mathbf{X}} \|[\mathbf{w}_2, r_1]\|_{\mathbf{Y}}.$$

This completes the proof.  $\square$

**Lemma 7.** *If HYPOTHESIS 2 is satisfied then, the operator  $\mathcal{A}_h : \mathbf{X}_h \rightarrow \mathbf{X}_h'$  defined by (19a) is continuous and  $\mathbf{X}_h$ -coercive on  $\mathbf{X}_h \cap \ker(\mathcal{B}_h)$  i.e.,*

$$\mathcal{A}_h[\mathbf{v}_1, q_2]([\mathbf{v}_1, q_2]) \geq C \|[\mathbf{v}_1, q_2]\|_{\mathbf{X}_h}^2, \quad \text{for all } [\mathbf{v}_1, q_2] \in \mathbf{X}_h \cap \ker(\mathcal{B}_h). \quad (22)$$

Where  $C > 0$  is an adequate constant depending only on the domain  $\Omega$ .

PROOF. The continuity of the operator  $\mathcal{A}_h$  follows from the continuity of the operator  $\mathcal{A}$ . For the coerciveness of the operator, let  $[\mathbf{v}_1, q_2] \in \mathbf{X}_h \cap \ker(\mathcal{B}_h)$  then

$$\mathcal{B}_h[\mathbf{v}_1, q_2]([\mathbf{w}_2, r_1]) = 0 \quad \text{for all } [\mathbf{w}_2, r_1] \in \mathbf{Y}_h. \quad (23)$$

Notice that  $\nabla \cdot \mathbf{v}_1|_L$  is constant for each  $L \in \mathcal{T}$  contained in  $\Omega_1$  and that  $\nabla q_2$  belongs to  $\nabla \mathcal{Q}(\Omega_2)$  by definition. Therefore,  $[\nabla q_2, \nabla \cdot \mathbf{v}_1] \in \mathbf{Y}_h$ , in particular, testing (23) with  $[0, r_1] \in \mathbf{Y}_h$  we conclude that  $\nabla \cdot \mathbf{v}_1 = 0$  since  $r_1$  is



an arbitrary element in  $L^2(\Omega_1)$ . On the other hand, clearly  $\nabla q_2 \in \nabla \mathcal{Q}(\Omega_2)$  and the pair  $[\nabla q_2, 0] \in \mathbf{Y}$  is eligible for testing (23); which yields  $\nabla q_2 = \mathbf{0}$ , i.e.  $q_2$  is constant inside  $\Omega_2$ . Hence

$$\int_{\Gamma} \beta q_2^2 = \frac{\|\beta \mathbf{1}_{\partial\Gamma}\|_{L^1(\Gamma)}}{|\Omega_2|} \|q_2\|_{0,\Omega_2}^2 = \frac{\|\beta \mathbf{1}_{\Gamma}\|_{L^1(\Gamma)}}{|\Omega_2|} \|q_2 \mathbf{1}_{\Omega_2}\|_{1,\Omega_2}^2.$$

Using the previous observations we get that

$$\begin{aligned} \mathcal{A}_h[\mathbf{v}_1, q_2](\mathbf{v}_1, q_2) &= \int_{\Omega_1} a \mathbf{v}_1 \cdot \mathbf{v}_1 + \int_{\Gamma} \beta q_2^2 dS \\ &\geq \left\| \frac{1}{a} \right\|_{L^\infty(\Omega)}^{-1} \|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 + \frac{\|\beta \mathbf{1}_{\Gamma}\|_{L^1(\Gamma)}}{|\Omega_2|} \|q_2\|_{1,\Omega_2}^2 \\ &\geq C \|\mathbf{v}_1, q_2\|_{\mathbf{X}}^2, \end{aligned}$$

where  $C = \min \left\{ \left\| \frac{1}{a} \right\|_{L^\infty(\Omega)}^{-1}, |\Omega_2|^{-1} \|\beta \mathbf{1}_{\Gamma}\|_{L^1(\Gamma)} \right\}$ . This completes the proof.  $\square$

**Theorem 8.** *Let  $\Omega$  be a polygonal region and let  $\mathcal{T}$  be a triangulation, then if the HYPOTHESIS 2 is satisfied, the PROBLEM (20) is well-posed.*

PROOF. It is direct to see that the operator  $\mathcal{C}_h$  is non-negative and symmetric. Due to this fact, LEMMA 6 and LEMMA 7, the hypotheses of THEOREM 1 are satisfied and the result follows.  $\square$

### 3.1. A Convergence Result

In this section we prove rigorously, under mild hypotheses on a sequence of triangulations  $\{\mathcal{T}^h : h > 0\}$ , the strong convergence of discrete solutions to the continuous one i.e.,  $(\mathbf{u}^h, p^h) \rightarrow (\mathbf{u}, p)$ , when  $h \rightarrow 0$ . In order to attain a-priori estimates some previous results are necessary.

**Proposition 9.** *Let  $\Omega$  be a domain satisfying HYPOTHESIS 1 then, there exists  $C > 0$  depending only on the map  $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$  such that*

$$\|\xi\|_{H^1(\Omega_i)}^2 \leq C^2 \left( \|\nabla \xi\|_{L^2(\Omega_i)}^2 + \|\sqrt{\beta} \xi\|_{L^2(\Gamma)}^2 \right), \quad \text{for all } \xi \in H^1(\Omega_i) \text{ and } i = 1, 2. \quad (24)$$

PROOF. Notice that the form  $\xi \mapsto \left( \|\nabla \xi\|_{L^2(\Omega_i)}^2 + \|\sqrt{\beta} \xi\|_{L^2(\Gamma)}^2 \right)^{1/2}$ , for if it is equal to zero it follows that  $\xi$  is constant, therefore

$$0 = \int_{\Gamma} \beta |\xi|^2 = |\xi|^2 \|\beta\|_{L^1(\Gamma)}.$$

Due to the HYPOTHESIS 2, this implies that  $\xi = 0$ . From here, a standard application of the Rellich-Kondrachov Theorem delivers the result.  $\square$

**Proposition 10.** *Let  $\mathcal{T}^h$  be a consistent triangulation of  $\Omega$  and let  $([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h]) \in \mathbf{X}_h \times \mathbf{Y}_h$  be the solution of PROBLEM (20), then there exists  $C > 0$  depending only on the domain  $\Omega$  such that*

$$\|p_1^h\|_{0,\Omega_1} \leq C \left( \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|p_2^h\|_{1,\Omega_2}^2 + \|\mathbf{g}\|_{1,\Omega_2}^2 + \|f_{\Sigma}\|_{1,\Omega_2}^2 \right)^{1/2}. \quad (25)$$

PROOF. Test PROBLEM (20) with  $([\mathbf{v}_1, 0], [\mathbf{0}, 0]) \in \mathbf{X}_h \times \mathbf{Y}_h$  and add both equations, this gives

$$\int_{\Omega_1} p_1^h \nabla \cdot \mathbf{v}_1 = - \int_{\Omega_1} a \mathbf{u}_1^h \cdot \mathbf{v}_1 - \int_{\Gamma} p_2^h (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS + \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}_1 - \int_{\Gamma} f_{\Sigma} (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS. \quad (26)$$

Applying the CBS inequality to each summand we get

$$\begin{aligned}
\left| \int_{\Omega_1} p_1 \nabla \cdot \mathbf{v}_1 \right| &\leq C (\|\mathbf{u}_1^h\|_{0,\Omega_1} \|\mathbf{v}_1\|_{0,\Omega_1} + \|p_2^h\|_{1/2,\Gamma} \|\mathbf{v}_1\|_{-1/2,\Gamma} + \|\mathbf{g}\|_{0,\Omega_1} \|\mathbf{v}_1\|_{0,\Omega_1} + \|f_\Sigma\|_{1/2,\Gamma} \|\mathbf{v}_1\|_{-1/2,\Gamma}) \\
&\leq C (\|\mathbf{u}_1^h\|_{0,\Omega_1} + \|p_2^h\|_{1,\Omega_2} + \|\mathbf{g}\|_{0,\Omega_1} + \|f_\Sigma\|_{1/2,\Gamma}) \|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \\
&\leq 2C (\|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|p_2^h\|_{1,\Omega_2}^2 + \|\mathbf{g}\|_{0,\Omega_1}^2 + \|f_\Sigma\|_{1/2,\Gamma}^2)^{\frac{1}{2}} \|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)}.
\end{aligned}$$

Here the generic constant of the second line is large enough. Due to THEOREM 5 there exists  $\mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1)$  such that  $\nabla \cdot \mathbf{v}_1 = p_1^h$  and  $\|\mathbf{v}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq C \|p_1^h\|_{0,\Omega_1}$ , where the generic bound  $C > 0$ , depends only on the domain  $\Omega_1$ . Testing the expression above with this function, the INEQUALITY (25) follows.  $\square$

Now we are ready to present an a-priori estimate.

**Theorem 11.** *Let  $\{\mathcal{T}^h : h > 0\}$  be a monotone sequence of consistent triangulations of  $\Omega$ . Denote by  $([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h]) \in \mathbf{X}_h \times \mathbf{Y}_h$  the solution of PROBLEM (20) associated to the triangulation  $\mathcal{T}^h$  with the fixed forcing terms  $F, \mathbf{g}, f_\Sigma, f_{\hat{n}}$ . Then, there exists  $C > 0$  such that*

$$\|[\mathbf{u}_1^h, p_2^h]\|_{\mathbf{X}} + \|[\mathbf{u}_2^h, p_1^h]\|_{\mathbf{Y}} \leq C, \quad \text{for all } h > 0. \quad (27)$$

PROOF. Test PROBLEM (20) with  $(\mathbf{u}^h, p^h)$  and add both equations, this gives

$$\int_{\Omega_1} a |\mathbf{u}_1^h|^2 + \int_{\Omega_2} a |\mathbf{u}_2^h|^2 + \int_{\Gamma} \beta |p_2^h|^2 dS = \int_{\Omega} F p^h - \int_{\Omega} \mathbf{g} \cdot \mathbf{u}^h + \int_{\Gamma} f_\Sigma (\mathbf{u}_1^h \cdot \hat{\mathbf{n}}) dS - \int_{\Gamma} f_{\hat{n}} p_2^h dS. \quad (28)$$

On the term in the right hand side, we apply first the usual duality bounds and next the CBS inequality for vectors in  $\mathbb{R}^4$ , this gives

$$\begin{aligned}
C_0 [\|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\mathbf{u}_2^h\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2] \\
\leq \|F\|_{0,\Omega} \|p^h\|_{0,\Omega} + \|\mathbf{g}\|_{L^2(\Omega)} \|\mathbf{u}^h\|_{0,\Omega} + \|f_\Sigma\|_{1/2,\Gamma} \|\mathbf{u}_1^h\|_{\mathbf{H}_{\text{div}}(\Omega_1)} + \|f_{\hat{n}}\|_{0,\Gamma} \|p_2^h\|_{0,\Gamma} \\
\leq \sqrt{2} [\|F\|_{0,\Omega}^2 + \|\mathbf{g}\|_{0,\Omega}^2 + \|f_\Sigma\|_{1/2,\Gamma}^2 + \|f_{\hat{n}}\|_{0,\Gamma}^2]^{1/2} \\
\left[ \|p_1^h\|_{0,\Omega_1}^2 + \|p_2^h\|_{0,\Gamma}^2 + \|\mathbf{u}_1^h\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 + \|\mathbf{u}_2^h\|_{0,\Omega_2}^2 \right]^{1/2}. \quad (29)
\end{aligned}$$

In the expression above, the constant  $\sqrt{2}$  appears due to the estimate  $\|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\mathbf{u}_1^h\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 \leq 2 \|\mathbf{u}_1^h\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2$ . Next, we focus on giving estimates to the second factor of the right hand side. In order to bound the pressure, first split it in two pieces  $\|p^h\|_{0,\Omega}^2 = \|p_1^h\|_{0,\Omega_1}^2 + \|p_2^h\|_{0,\Omega_2}^2$ , now due to PROPOSITION 9, there exists  $C > 0$  depending only on the map  $\mathcal{G}$  such that

$$\begin{aligned}
\frac{1}{C} \|p_1^h\|_{0,\Omega_1}^2 &\leq \|\nabla p_1^h\|_{0,\Omega_1}^2 + \|\sqrt{\beta} p_1^h\|_{0,\Gamma}^2 \\
&\leq 2 \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + 2 \|\mathbf{g}^h\|_{0,\Omega_1}^2 + 2 \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 + 2 \|\sqrt{\beta} f_\Sigma\|_{0,\Gamma}^2 \\
&\leq 2 \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + 2 \|\mathbf{g}\|_{0,\Omega_1}^2 + 2 \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 + 2 \|\sqrt{\beta}\|_{L^\infty(\Gamma)} \|f_\Sigma\|_{0,\Gamma}^2.
\end{aligned} \quad (30)$$

The second inequality holds due to the strong discretized Darcy equation (6a) i.e.  $\mathbf{u}_1^h + \nabla p_1^h = \mathbf{g}^h$ , with  $\mathbf{g}^h$  denoting the orthogonal projection of  $\mathbf{g}$  onto  $\nabla \mathcal{Q}(\Omega_2, \mathcal{T}^h)$ . In addition,  $\|\mathbf{g}^h\|_{0,\Omega_1} \leq \|\mathbf{g}\|_{0,\Omega_1}$  which gives the third inequality. On the other hand, combining the estimates (25) and (24) with (30) gives

$$\|p_1^h\|_{0,\Omega_1}^2 \leq C (\|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 + \|\mathbf{g}\|_{0,\Omega_1}^2 + \|f_\Sigma\|_{1/2,\Gamma}^2), \quad (31)$$

for  $C > 0$  large enough depending only on  $\Omega$ . Next, due to (6b) it holds that  $\|\mathbf{u}_1^h\|_{\mathbf{H}_{\text{div}}(\Omega_1)}^2 = \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|F\|_{0,\Omega_1}^2$ . Denoting  $\kappa = \max\{1, 2\|\sqrt{\beta}\|_{L^\infty(\Gamma)}\}(\|F\|_{0,\Omega}^2 + \|\mathbf{g}\|_{0,\Omega}^2 + \|\mathbf{f}_\Sigma\|_{0,\Gamma}^2 + \|\mathbf{f}_\mathbf{h}\|_{1/2,\Gamma}^2)$  and introducing the previous observations in (29) we get

$$C_0^2 \left[ \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\mathbf{u}_2^h\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 \right]^2 \leq 2 \max\{C, 1\} \kappa \left[ \|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\mathbf{u}_2^h\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 \right] + \kappa^2.$$

The expression above shows that for all  $h > 0$ , a square function is controlled by a linear function of the same argument, therefore, there must exist yet another constant still denoted by  $C > 0$ , such that

$$\|\mathbf{u}_1^h\|_{0,\Omega_1}^2 + \|\mathbf{u}_2^h\|_{0,\Omega_2}^2 + \|\sqrt{\beta} p_2^h\|_{0,\Gamma}^2 \leq C, \quad \text{for all } h > 0.$$

From here, the strong Darcy equation (6d), the INEQUALITY (24), the INEQUALITY (31) and the conservation STATEMENT (6b) give the result.  $\square$

From the standard theory of general Hilbert spaces the following result is trivial.

**Corollary 12.** *Assuming the hypotheses of THEOREM 11 hold, there exist an element  $([\mathbf{u}'_1, p'_2], [\mathbf{u}'_2, p'_1]) \in \mathbf{X} \times \mathbf{Y}$  and a subsequence, still denoted the same, such that  $\{([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h]) : h > 0\}$ , which is weakly convergent to it.*

Before proving the convergence statement we recall a standard finite element result.

**Proposition 13.** *Let  $\Omega$  be an open polygonal domain of  $\mathbb{R}^2$  satisfying HYPOTHESIS 1 and let  $\{\mathcal{T}^h : h > 0\}$  be a monotone sequence of consistent triangulations with size  $h \rightarrow 0$ , then*

$$\text{cl} \{ \mathbf{RT}_0(\Omega_1, \mathcal{T}^h) : h > 0 \} = \mathbf{H}_{\text{div}}(\Omega_1), \quad (32a)$$

$$\text{cl} \{ \mathcal{Q}(\Omega_2, \mathcal{T}^h) : h > 0 \} = E(\Omega_2), \quad (32b)$$

$$\text{cl} \{ \nabla \mathcal{Q}(\Omega_2, \mathcal{T}^h) : h > 0 \} = \mathbf{V}(\Omega_2), \quad (32c)$$

$$\text{cl} \{ \mathcal{Q}(\Omega_1, \mathcal{T}^h) : h > 0 \} = L^2(\Omega_2). \quad (32d)$$

PROOF. The identities (32a) and (32d) are standard conformal finite element results. For the identity (32b) it is enough to extend, in a continuous and linear fashion, the elements of  $\mathcal{Q}(\Omega_2, \mathcal{T}^h)$  to polynomials of degree one in the whole domain  $\Omega$ . This extension yields the classic FEM space of continuous, piecewise linear affine functions on the whole domain  $\Omega$ , associated to  $\mathcal{T}^h$ , which we denote by  $\mathcal{Q}^1(\Omega, \mathcal{T}^h)$ . From the standard theory of conformal finite elements, we know that  $\text{cl} \{ \mathcal{Q}^1(\Omega, \mathcal{T}^h) : h > 0 \} = H^1(\Omega)$ , in particular, the statement (32b) holds. Finally, the identity (32c) follows trivially from (32b).

Next we prove the convergence of the solutions and identify the limiting problem.

**Theorem 14.** *Let  $\{\mathcal{T}^h : h > 0\}$ ,  $F$ ,  $\mathbf{g}$ ,  $\mathbf{f}_\Sigma$ ,  $\mathbf{f}_\mathbf{h}$ ,  $([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h])$  be as in THEOREM 11. Then, the element  $([\mathbf{u}'_1, p'_2], [\mathbf{u}'_2, p'_1])$  given by COROLLARY 12 is the unique solution to PROBLEM (16). Moreover, the whole sequence converges to this point i.e.,*

$$([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h]) \xrightarrow{h \rightarrow 0} ([\mathbf{u}_1, p_2], [\mathbf{u}_2, p_1]), \quad \text{weakly in } \mathbf{X} \times \mathbf{Y}. \quad (33)$$

PROOF. In order to prove the result, it is enough to show that  $([\mathbf{u}'_1, p'_2], [\mathbf{u}'_2, p'_1])$  satisfies the variational STATEMENT (14). Let  $([\mathbf{v}_1, q_2], [\mathbf{v}_2, q_1])$  be an arbitrary element of  $\mathbf{X} \times \mathbf{Y}$  and let  $([\mathbf{v}_1^h, q_2^h], [\mathbf{v}_2^h, q_1^h])$  be its orthogonal

projection onto  $\mathbf{X}_h \times \mathbf{Y}_h$ . Due to PROPOSITION 13 the sequence  $\{([\mathbf{v}_1^h, q_2^h], [\mathbf{v}_2^h, q_1^h]) : h > 0\}$  converges strongly to  $([\mathbf{v}_1, q_2], [\mathbf{v}_2, q_1])$ . Now test the variational formulation associated to PROBLEM (20), this gives

$$\begin{aligned} & \int_{\Omega_1} a \mathbf{u}_1^h \cdot \mathbf{v}_1^h + \int_{\Gamma} \beta p_2^h q_2^h dS - \int_{\Gamma} (\mathbf{u}_1^h \cdot \hat{\mathbf{n}}) q_2^h dS + \int_{\Gamma} p_2^h (\mathbf{v}_1^h \cdot \hat{\mathbf{n}}) dS \\ & - \int_{\Omega_1} p_1^h \nabla \cdot \mathbf{v}_1^h - \int_{\Omega_2} \mathbf{u}_2^h \cdot \nabla q_2^h = \int_{\Omega_2} F q_2^h - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}_1^h + \int_{\Gamma} f_{\Sigma} (\mathbf{v}_1^h \cdot \hat{\mathbf{n}}) dS - \int_{\Gamma} f_{\hat{\mathbf{n}}} q_2^h dS, \\ & \int_{\Omega_1} \nabla \cdot \mathbf{u}_1^h q_1^h + \int_{\Omega_2} \nabla p_2^h \cdot \mathbf{v}_2^h + \int_{\Omega_2} a \mathbf{u}_2^h \cdot \mathbf{v}_2^h = \int_{\Omega_1} F q_1^h - \int_{\Omega_2} \mathbf{g} \cdot \mathbf{v}_2^h. \end{aligned}$$

Notice that in both expressions above each summand of the right hand side converges since one of the factors is weakly convergent, while the other is strongly convergent. The right hand side also converges due to the strong convergence of the quantifiers. Consequently, the element  $([\mathbf{u}_1', p_2'], [\mathbf{u}_2', p_1'])$  satisfies the variational STATEMENT (14) for any arbitrary test function  $([\mathbf{v}_1, q_2], [\mathbf{v}_2, q_1])$ . It follows that  $([\mathbf{u}_1', p_2'], [\mathbf{u}_2', p_1'])$  is a solution of PROBLEM (16), this concludes the first part of the theorem.

For the second part, the well-posedness of PROBLEM (16) gives the uniqueness of its solution. Consequently, due to the ESTIMATE (27), any subsequence of  $\{([\mathbf{v}_1^h, q_2^h], [\mathbf{v}_2^h, q_1^h]) : h > 0\}$  would have yet another subsequence weakly convergent to the solution of PROBLEM (16). Hence, the STATEMENT (33) follows and the proof is complete.  $\square$

Finally, we have

**Theorem 15.** *Let  $\{\mathcal{T}^h : h > 0\}$ ,  $F$ ,  $\mathbf{g}$ ,  $f_{\Sigma}$ ,  $f_{\hat{\mathbf{n}}}$ ,  $([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h])$  be as in THEOREM 11 above and let  $([\mathbf{u}_1, p_2], [\mathbf{u}_2, p_1])$  be the solution to PROBLEM (16). Then*

$$([\mathbf{u}_1^h, p_2^h], [\mathbf{u}_2^h, p_1^h]) \xrightarrow{h \rightarrow 0} ([\mathbf{u}_1, p_2], [\mathbf{u}_2, p_1]), \quad \text{strongly in } \mathbf{X} \times \mathbf{Y}. \quad (35)$$

PROOF. We use the standard approach. Test PROBLEM (16) with  $(\mathbf{u}, p)$  and add both equations, this yields

$$\int_{\Omega_1} a |\mathbf{u}_1|^2 + \int_{\Omega_2} a |\mathbf{u}_2|^2 + \int_{\Gamma} \beta |p_2|^2 dS = \int_{\Omega} F p - \int_{\Omega} \mathbf{g} \cdot \mathbf{u} + \int_{\Gamma} f_{\Sigma} (\mathbf{u}_1 \cdot \hat{\mathbf{n}}) dS - \int_{\Gamma} f_{\hat{\mathbf{n}}} p_2 dS. \quad (36)$$

On the other hand, taking lim sup in the IDENTITY (28) we get

$$\begin{aligned} \limsup_{h \rightarrow 0} \left[ \int_{\Omega_1} a |\mathbf{u}_1^h|^2 + \int_{\Omega_2} a |\mathbf{u}_2^h|^2 + \int_{\Gamma} \beta |p_2^h|^2 dS \right] &= \int_{\Omega} F p - \int_{\Omega} \mathbf{g} \cdot \mathbf{u} + \int_{\Gamma} f_{\Sigma} (\mathbf{u}_1 \cdot \hat{\mathbf{n}}) dS - \int_{\Gamma} f_{\hat{\mathbf{n}}} p_2 dS \\ &= \int_{\Omega_1} a |\mathbf{u}_1|^2 + \int_{\Omega_2} a |\mathbf{u}_2|^2 + \int_{\Gamma} \beta |p_2|^2 dS \\ &\leq \liminf_{h \rightarrow 0} \left[ \int_{\Omega_1} a |\mathbf{u}_1^h|^2 + \int_{\Omega_2} a |\mathbf{u}_2^h|^2 + \int_{\Gamma} \beta |p_2^h|^2 dS \right]. \end{aligned}$$

In the expression above the equality of the second line holds due to the IDENTITY (36) and the inequality of the third line holds due to the weak convergence STATEMENT (33). From here, due to standard Hilbert space theory, it follows that

$$\|\mathbf{u}_1^h - \mathbf{u}_1\|_{0,\Omega} \xrightarrow{h \rightarrow 0} 0, \quad (37a)$$

$$\|\mathbf{u}_2^h - \mathbf{u}_2\|_{0,\Omega} \xrightarrow{h \rightarrow 0} 0, \quad (37b)$$

$$\|\sqrt{\beta}(p_2^h - p_2)\|_{0,\Gamma} \xrightarrow{h \rightarrow 0} 0. \quad (37c)$$

On the other hand, the solution  $\mathbf{u}_1^h$  satisfies the discretization of the EQUATION (6b). Therefore, it holds that  $\nabla \cdot \mathbf{u}_1^h = F^h$ , where  $F^h$  is the orthogonal projection of  $F$  on the space  $\mathcal{Q}(\Omega_1, \mathcal{T}^h)$ . Since  $\|F^h - F\|_{0,\Omega_1} \rightarrow 0$  it follows that  $\|\nabla \cdot \mathbf{u}_1^h - \nabla \cdot \mathbf{u}_1\|_{0,\Omega_1} \rightarrow 0$  which, combined with the STATEMENT (37a) yields

$$\|\mathbf{u}_1^h - \mathbf{u}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \xrightarrow{h \rightarrow 0} 0. \quad (38)$$

Next, the solution  $p_2^h$  satisfies the discretized version of Darcy's law (6d) i.e.,  $\mathbf{u}_2^h + \nabla p_2^h = \mathbf{g}^h$ . Again,  $\mathbf{g}^h$  indicates the orthogonal projection of  $\mathbf{g}$  onto  $\nabla \mathcal{Q}(\Omega_2, \mathcal{T}^h)$  and due to the strong convergence of the orthogonal projections it follows that  $\|\nabla p_2^h - \nabla p_2\|_{0,\Omega_2} \rightarrow 0$ . The latter, combined with the STATEMENT (37c) and the INEQUALITY (24) implies

$$\|p_2^h - p_2\|_{1,\Omega_2} \xrightarrow{h \rightarrow 0} 0. \quad (39)$$

Finally, for the strong convergence of  $\{p_1^h : h > 0\}$ , test the STATEMENT (14a) with  $[\mathbf{v}_1, 0] \in \mathbf{X}_h$  to get

$$\int_{\Omega_1} p_1 \nabla \cdot \mathbf{v}_1 = - \int_{\Omega_1} a \mathbf{u}_1^h \cdot \mathbf{v}_1 - \int_{\Gamma} p_2^h (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS + \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}_1 - \int_{\Gamma} f_{\Sigma} (\mathbf{v}_1 \cdot \hat{\mathbf{n}}) dS.$$

Equating the expression above with the IDENTITY (26) we get

$$\int_{\Omega_1} p_1^h \nabla \cdot \mathbf{v}_1 = \int_{\Omega_1} p_1 \nabla \cdot \mathbf{v}_1.$$

The above holds for all  $\mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1)$  in particular, taking the supremum in the expression above and recalling THEOREM 5 we get

$$\begin{aligned} \|p_1^h\|_{0,\Omega_1} &\leq \sup \left\{ \int_{\Omega_1} p_1^h \nabla \cdot \mathbf{v}_1 : \mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1), \|\nabla \cdot \mathbf{v}_1\|_{0,\Omega_1} = 1 \right\} \\ &\leq \sup \left\{ \int_{\Omega_1} p_1 \nabla \cdot \mathbf{v}_1 : \mathbf{v}_1 \in \mathbf{RT}_0(\Omega_1), \|\nabla \cdot \mathbf{v}_1\|_{0,\Omega_1} = 1 \right\} \\ &\leq \sup \left\{ \int_{\Omega_1} p_1 \nabla \cdot \mathbf{w}_1 : \mathbf{w}_1 \in \mathbf{H}_{\text{div}}(\Omega_1), \|\nabla \cdot \mathbf{w}_1\|_{0,\Omega_1} = 1 \right\} \\ &\leq \|p_1\|_{0,\Omega_1}. \end{aligned}$$

The last inequality in the expression above holds from the general theory of Hilbert spaces. Thus,  $\limsup \|p_1^h\|_{0,\Omega_1} \leq \|p_1\|_{0,\Omega_1}$  and due to the weak convergence it also holds that  $\|p_1\|_{0,\Omega_1} \leq \liminf \|p_1^h\|_{0,\Omega_1}$ . Therefore the norms converge and it follows that

$$\|p_1^h - p_1\|_{0,\Omega_1} \xrightarrow{h \rightarrow 0} 0.$$

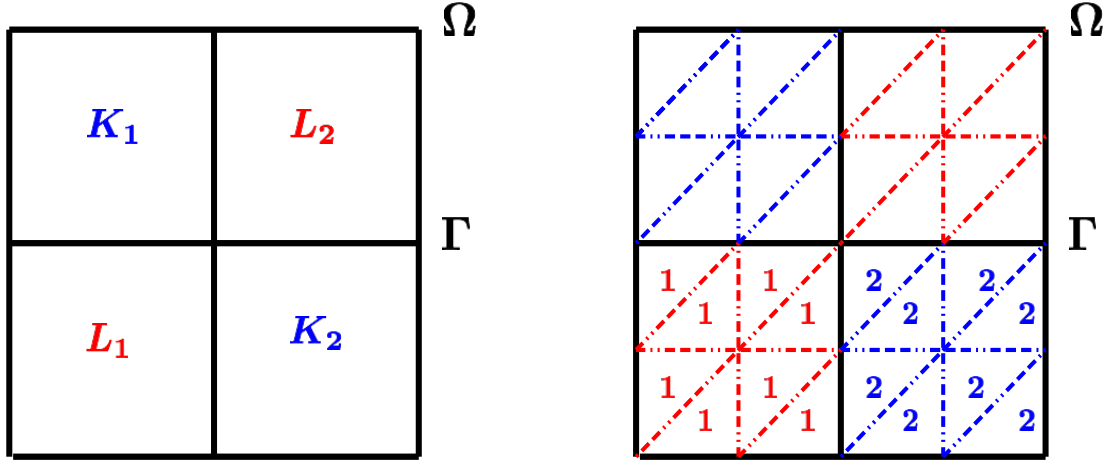
This concludes the proof.  $\square$

#### 4. Numerical Examples

In this section we present two numerical examples to illustrate the method, the first showing a case of continuity, the second a slight perturbation of the first to illustrate how the method handles discontinuities across interfaces. The numerical examples use the finite dimensional spaces  $\mathbf{X}_h, \mathbf{Y}_h$  introduced in (18). The experiments are executed in a MATLAB script using adaptations of the codes **EBmfem.m** (see, [21], [22]) and **fem2d.m** (see, [23], [24]).

For the sake of clarity, we adopt the domain  $\Omega$ , the interface  $\Gamma$  and the subdomains  $\Omega_1, \Omega_2$  as follows (see FIGURE 2 (a))

$$\begin{aligned} \Omega &\stackrel{\text{def}}{=} (-1, 1) \times (-1, 1), & \Gamma &\stackrel{\text{def}}{=} (-1, 1) \times \{0\} \cup \{0\} \times (-1, 1), \\ \Omega_1 &\stackrel{\text{def}}{=} (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1), & \Omega_2 &\stackrel{\text{def}}{=} (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0). \end{aligned} \quad (40)$$



(a) Basic experimentation domain  $\Omega$  and map  $\mathcal{G}$ .

(b) Consistent grid example.

Figure 2: Figure (a) depicts a bipartite map  $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$  example for the region  $\Omega = [0, 1] \times [0, 1]$ . Subregions belonging to  $\mathcal{G}_1$  are red-colored and the subregions belonging to  $\mathcal{G}_2$  are blue-colored. Figure (b) depicts an example of a grid  $\mathcal{T}$  consistent with the map  $\mathcal{G}$ . Some of the triangles belonging to  $\mathcal{T}_1$  and  $\mathcal{T}_2$  have been labeled with 1 and 2 respectively.

Again, for simplicity, all the experiments run on the uniform Cartesian grid, see FIGURE 2 (b). The sequence of grids  $\{\mathcal{T}^i : 0 \leq i \leq 5\}$  has corresponding sizes  $h_i = \frac{1}{2^i}$  for  $0 \leq i \leq 5$ ; consequently it is a monotone sequence as described in DEFINITION 3. The experimental computation for the order of convergence  $r$ , uses the standard approach. Assuming that the error satisfies  $e = \mathcal{O}(h^r)$ , we approximate  $r$  by

$$r \sim \frac{\log e_{k+1} - \log e_k}{\log h_{k+1} - \log h_k} = \frac{\log e_k - \log e_{k+1}}{\log 2}, \quad \text{for all } 0 \leq k \leq 4.$$

In the expression above, the last equality holds due to the particular nature of the grids' size.

**Example 1.** The purpose of the present example is to illustrate how the method handles problems free of discontinuities across the interfaces. The exact solution in this case is given by

$$p : \Omega \rightarrow \mathbb{R}, \quad p(x, y) = x y (x - 1)^2 (y - 1)^2 (x + 1)^2 (y + 1)^2, \quad (41a)$$

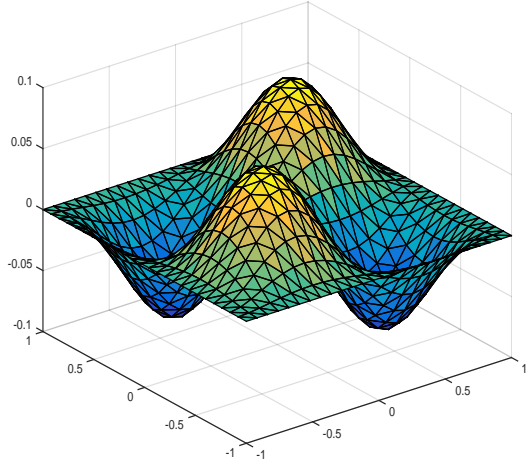
$$\mathbf{u} : \Omega \rightarrow \mathbb{R}^2, \quad \mathbf{u}(x, y) = -\nabla p(x, y), \quad (41b)$$

see FIGURE (3). The forcing terms are

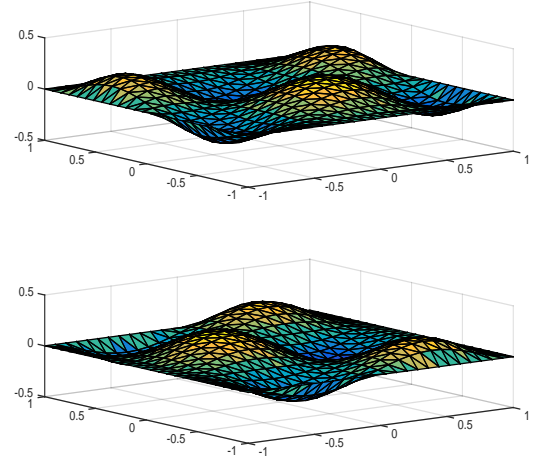
$$\begin{aligned} \mathbf{g} : \Omega &\rightarrow \mathbb{R}^2, & \mathbf{g} &= \mathbf{0}, \\ F : \Omega &\rightarrow \mathbb{R}, & F &= -\nabla \cdot \nabla p, \end{aligned} \quad (42a)$$

$$f_{\Sigma}, f_{\tilde{\mathbf{n}}} : \Gamma \rightarrow \mathbb{R}, \quad f_{\Sigma} = 0, \quad f_{\tilde{\mathbf{n}}} = 0. \quad (42b)$$

It is direct to see that  $[\mathbf{u}, p]$  defined by (41) is the exact solution of the PROBLEM (6) on the geometric domain described by (40) with the forcing terms defined in (42). In particular, the boundary conditions (6c), (6f) and the interface exchange conditions (7b), (7a) are satisfied.



(a) Pressure Exact Solution.



(b) Flux Exact Solution.

Figure 3: Figure (a) depicts the pressure of the exact solution  $p(x, y) = xy(x-1)^2(y-1)^2(x+1)^2(y+1)^2$ , see EQUATION (41a). Figure (b) depicts the flux of the exact solution  $\mathbf{u} = -\nabla p$ , see EQUATION (41b). On the upper right corner is depicted the  $x$ -component while the lower right corner displays the  $y$ -component.

The convergence results are displayed in the TABLES 1 and 2 below, the convergence rate behaves as expected, i.e.,

$$\begin{aligned} \|p_1^h - p_1\|_{0,\Omega_1} &= \mathcal{O}(h^{1.8}), & \|p_2^h - p_2\|_{0,\Omega_1} &= \mathcal{O}(h^2), & \|p_2^h - p_1\|_{1,\Omega_2} &= \mathcal{O}(h). \\ \|\mathbf{u}_1^h - \mathbf{u}_1\|_{0,\Omega_1} &= \mathcal{O}(h), & \|\mathbf{u}_1^h - \mathbf{u}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} &= \mathcal{O}(h), & \|\mathbf{u}_2^h - \mathbf{u}_2\|_{0,\Omega_2} &= \mathcal{O}(h). \end{aligned}$$

Finally, the numerical solution for  $h = \frac{1}{2^3}$  is depicted in FIGURE (4); the choice of the grid was based on optical clarity to illustrate both: the nature of discrete solution and its convergence to the continuous solution.

Table 1: Pressures Convergence Table

$h$	$\ p_1^h - p_1\ _{0,\Omega_1}$	$r$	$\ p_2^h - p_2\ _{0,\Omega_2}$	$r$	$\ p_1^h - p_1\ _{1,\Omega_2}$	$r$
1	0.1836		2.8144		0.8643	
$\frac{1}{2}$	0.0261	0.4383	0.0721	5.2867	0.1976	2.1289
$\frac{1}{4}$	0.0091	1.5201	0.0226	1.6737	0.0887	1.1556
$\frac{1}{8}$	0.0026	1.8074	0.0062	1.8660	0.0422	1.0717
$\frac{1}{16}$	0.0007	1.8931	0.0016	1.9542	0.0209	1.0137
$\frac{1}{32}$	0.0002	1.807	0.0004	2.0000	0.0104	1.0069

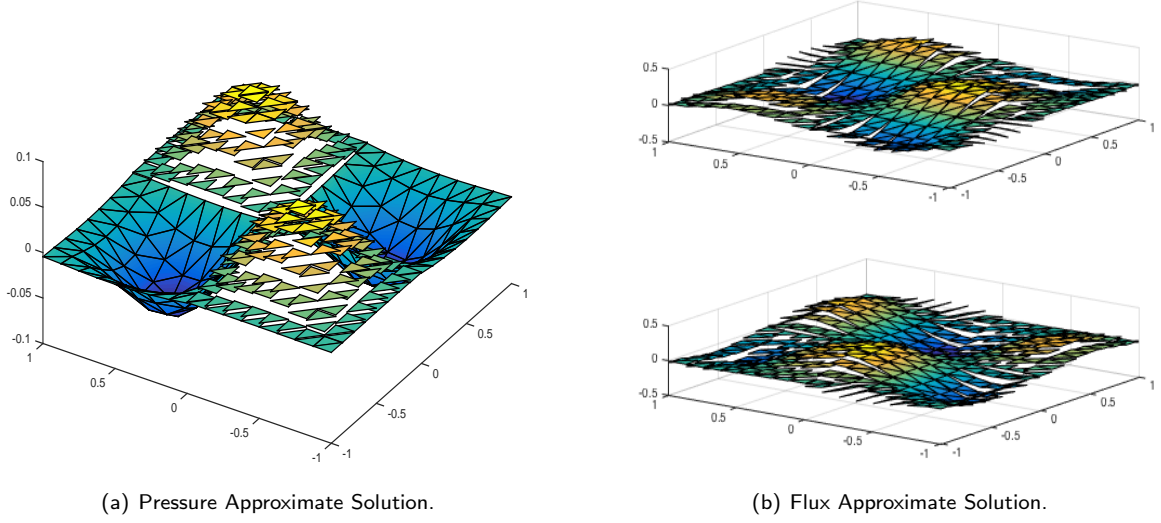


Figure 4: Solution for a mesh of size  $h = \frac{1}{8}$ , the sub-domains are  $\Omega_1 = (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1)$  and  $\Omega_2 = (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0)$ , see IDENTITY (40). Figure (a) depicts the pressure  $p^h$  of the approximate solution, it is piecewise constant on the domain  $\Omega_1$  and piecewise linear affine on the domain  $\Omega_2$ . Figure (b) depicts the flux of the approximate solution  $\mathbf{u}^h$ . On the upper right corner is depicted the  $x$ -component of the flux, which is continuous across **horizontal edges** of  $\Omega_1$  and piecewise constant on the domain  $\Omega_2$ . On the lower right corner we display the  $y$ -component of the flux, which is continuous across **vertical edges** of  $\Omega_1$  and piecewise constant on the domain  $\Omega_2$ .

Table 2: Velocities Convergence Table

$h$	$\ \mathbf{u}_1^h - \mathbf{u}_1\ _{0,\Omega_1}$	$r$	$\ \mathbf{u}_1^h - \mathbf{u}_1\ _{\mathbf{H}_{\text{div}}(\Omega_1)}$	$r$	$\ \mathbf{u}_2^h - \mathbf{u}_2\ _{0,\Omega_2}$	$r$
1	0.8264		0.8264		0.9184	
$\frac{1}{2}$	0.1409	2.5522	0.1409	2.5522	0.1840	2.3194
$\frac{1}{4}$	0.0617	1.1913	0.0617	1.1913	0.0857	1.1023
$\frac{1}{8}$	0.0302	1.0307	0.0302	1.0307	0.0417	1.0392
$\frac{1}{16}$	0.0150	1.0096	0.0150	1.0096	0.0208	1.0035
$\frac{1}{32}$	0.0075	1.0000	0.0075	1.0000	0.0104	1.0000

**Example 2.** The present example is a perturbation of the previous one, in order to illustrate how the method handles problems with simultaneous discontinuities across the interfaces in both: the normal flux and the normal stress. The perturbation is localized on the fourth quadrant of the domain  $(0, 1) \times (0, -1)$ . The analytic solution in this case is given by

$$p : \Omega \rightarrow \mathbb{R}, \quad (43a)$$

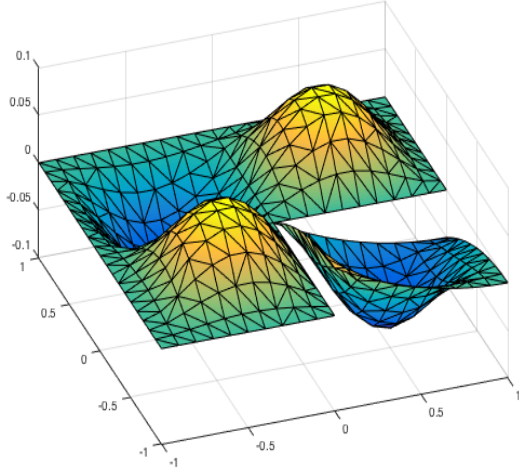
$$p(x, y) = xy(x-1)^2(y-1)^2(x+1)^2(y+1)^2 + \frac{1}{20}((x-1)^2 - (y+1)^2)\mathbb{1}_{(1,0) \times (0,-1)}(x, y),$$

$$\mathbf{u} : \Omega \rightarrow \mathbb{R}^2, \quad \mathbf{u}(x, y) = -\nabla p(x, y), \quad (43b)$$

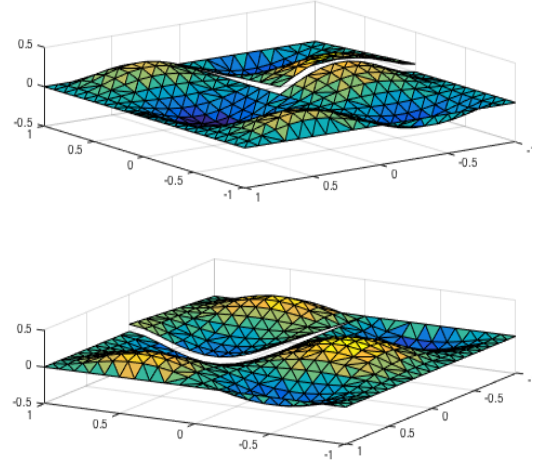
see FIGURE (5). The forcing terms are acting inside the domains are identical to the previous example,

$$\begin{aligned} \mathbf{g} : \Omega &\rightarrow \mathbb{R}^2, & \mathbf{g} &= \mathbf{0}, \\ F : \Omega &\rightarrow \mathbb{R}, & F &= -\nabla \cdot \nabla p, \end{aligned} \quad (44a)$$





(a) Discontinuous Pressure Exact Solution.



(b) Discontinuous Flux Exact Solution.

Figure 5: Discontinuous exact solution, the visualization angles are different for the pressure and the flux; the choice is made focusing on the jumps of discontinuity. The discontinuities take place on the interface subset  $\{0\} \times (-1, 0) \cup (0, 1) \times \{0\}$ . Figure (a) depicts the pressure of the exact solution  $p$ , see EQUATION (43a). Figure (b) depicts the flux of the exact solution  $\mathbf{u} = -\nabla p$ , see EQUATION (43b). On the upper right corner is depicted the x-component while the lower right corner displays the y-component.

In this case, the interface forcing terms account for the jumps of the solution across the interface, i.e., according to the interface exchange conditions (7b), (7a),  $f_{\Sigma}$  and  $f_{\hat{n}}$  are given by

$$\begin{aligned}
 f_{\Sigma} &: \Gamma \rightarrow \mathbb{R}, \\
 f_{\Sigma}(x, y) &= \frac{1}{20} ((x-1)^2 - 1) \mathbb{1}_{(0,1) \times \{0\}}(x, y) + \frac{1}{20} (1 - (y+1)^2) \mathbb{1}_{\{0\} \times (-1,0)}(x, y), \\
 f_{\hat{n}} &: \Gamma \rightarrow \mathbb{R}, \\
 f_{\hat{n}}(x, y) &= \frac{1}{20} (x-4) \mathbb{1}_{(0,1) \times \{0\}}(x, y) + \frac{1}{20} (4-y) \mathbb{1}_{\{0\} \times (-1,0)}(x, y).
 \end{aligned} \tag{44b}$$

It is direct to see that  $[\mathbf{u}, p]$  defined by (43) is the exact solution to the PROBLEM (6) on the geometric

Table 3: Pressures Convergence Table

$h$	$\ p_1^h - p_1\ _{0,\Omega_1}$	$r$	$\ p_2^h - p_2\ _{0,\Omega_2}$	$r$	$\ p_1^h - p_1\ _{1,\Omega_2}$	$r$
1	0.9984		1.6520		2.5140	
$\frac{1}{2}$	0.0261	5.2575	0.0721	4.5181	0.1980	3.6664
$\frac{1}{4}$	0.0091	1.5201	0.0226	1.6737	0.0889	1.1552
$\frac{1}{8}$	0.0026	1.8074	0.0062	1.8660	0.0423	1.0715
$\frac{1}{16}$	0.0007	1.8931	0.0016	1.9542	0.0209	1.0172
$\frac{1}{32}$	0.0002	1.8074	0.0004	2.0000	0.0104	1.0069

domain described by (40) with the forcing terms defined in (44). Again, the boundary and interface conditions are satisfied.

The convergence results are displayed in the TABLES 3 and 4 below. The convergence behavior is virtually identical to the continuous case with observable differences (TABLES 1 and 2) only for the first stages.

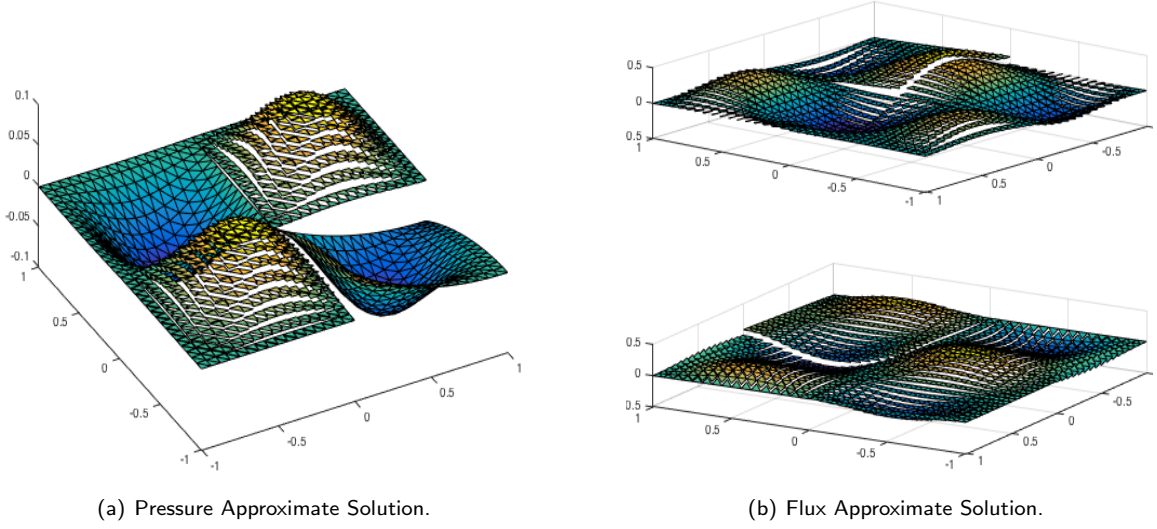


Figure 6: Solution for a mesh of size  $h = \frac{1}{16}$ , the visualization angles are different for the pressure and the flux; the choice is made focusing on the jumps of discontinuity. The discontinuities take place on the interface subset  $\{0\} \times (-1, 0) \cup (0, 1) \times \{0\}$ . Figure (a) depicts the pressure  $p^h$  of the approximate solution, it is piecewise constant on the domain  $\Omega_1$  and piecewise linear affine on the domain  $\Omega_2$ . Figure (b) depicts the flux of the approximate solution  $\mathbf{u}^h$ . On the upper right corner is depicted the x-component of the flux, which is continuous across **horizontal edges** of  $\Omega_1$  and piecewise constant on the domain  $\Omega_2$ . On the lower right corner we display the y-component of the flux, which is continuous across **vertical edges** of  $\Omega_1$  and piecewise constant on the domain  $\Omega_2$ .

Table 4: Velocities Convergence Table

$h$	$\ \mathbf{u}_1^h - \mathbf{u}_1\ _{0,\Omega_1}$	$r$	$\ \mathbf{u}_1^h - \mathbf{u}_1\ _{\mathbf{H}_{\text{div}}(\Omega_1)}$	$r$	$\ \mathbf{u}_2^h - \mathbf{u}_2\ _{0,\Omega_2}$	$r$
1	19.8825		19.8825		9.8017	
$\frac{1}{2}$	0.1409	7.1407	0.1409	7.1407	0.1844	5.7321
$\frac{1}{4}$	0.0617	1.1913	0.0617	1.1913	0.0860	1.1004
$\frac{1}{8}$	0.0302	1.0307	0.0302	1.0307	0.0418	1.0408
$\frac{1}{16}$	0.0150	1.0096	0.0150	1.0096	0.0208	1.0069
$\frac{1}{32}$	0.0075	1.0000	0.0075	1.0000	0.0104	1.0000

Consequently, the convergence rate behaves as expected, i.e.,

$$\begin{aligned}
\|p_1^h - p_1\|_{0,\Omega_1} &= \mathcal{O}(h^{1.8}), & \|p_2^h - p_2\|_{0,\Omega_1} &= \mathcal{O}(h^2), & \|p_2^h - p_1\|_{1,\Omega_2} &= \mathcal{O}(h). \\
\|\mathbf{u}_1^h - \mathbf{u}_1\|_{0,\Omega_1} &= \mathcal{O}(h), & \|\mathbf{u}_1^h - \mathbf{u}_1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} &= \mathcal{O}(h), & \|\mathbf{u}_2^h - \mathbf{u}_2\|_{0,\Omega_2} &= \mathcal{O}(h).
\end{aligned}$$

Finally, the numerical solution for  $h = \frac{1}{2^4}$  is depicted in FIGURE (6); the choice of the grid was based on optical clarity of the jumps across the interface.

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